Avoiding Coincidences

by Florian Frick

The complete graph on five vertices is non-planar that is, for any drawing in the plane, two edges share a point outside of a common vertex. The real projective plane $\mathbb{R}P^2$ does not embed into $\mathbb{R}^3$, that is, any continuous map $f : \mathbb{R}P^2 \longrightarrow \mathbb{R}^3$ has a double point. In fact, any immersion of $\mathbb{R}P^2$ into $\mathbb{R}^3$ has a triple point. This article discusses some fragments of the story of developing a theory of non-embeddability and extensions to higher multiplicity coincidences (such as the existence of triple points, quadruple points, etc.) from the viewpoint of discrete geometry.

For real numbers $x_0 \leq x_1 \leq \ldots \leq x_{2n}$ the set $\bigcap_{i=0}^{n} [x_i, x_{2n-i}]$ consists only of the point x_n — the *median* of the x_i, which leaves precisely n of the x_i to either side of it. Bryan Birch, of Birch and Swinnerton-Dyer fame, showed in a 1959 paper that one may observe a similar phenomenon for tuples of real numbers: Any $3n$ points in the plane $\mathbb{R}^2$ can be split into n triples of points such that the corresponding n triangles all have a point in common.

For higher dimensions, Birch asked whether any $(d+1)n$ points X in $\mathbb{R}^d$ determine n vertex-disjoint simplices of dimension d that all contain a common point of intersection. Any such point of intersection c would be a *center point* for X: Any half-space containing c also contains at least a fraction of $\frac{1}{d+1}$ of the points of X, as it must contain at least one vertex per simplex. For $d = 1$ this recovers the notion of median. Birch already recognized that one can be more economical about the number of points. He conjectured that any $(n-1)(d+1)+1$ points in $\mathbb{R}^d$ may be partitioned into n sets $X_1, \ldots, X_n$ whose convex hulls contain a common point and confirmed this conjecture for $d = 2$. It is easily checked that for a sufficiently generic set of $(n-1)(d+1)$ points in $\mathbb{R}^d$ even the affine hulls of any partition $X_1, \ldots, X_n$ into n parts have empty intersection, and thus this result would be optimal.

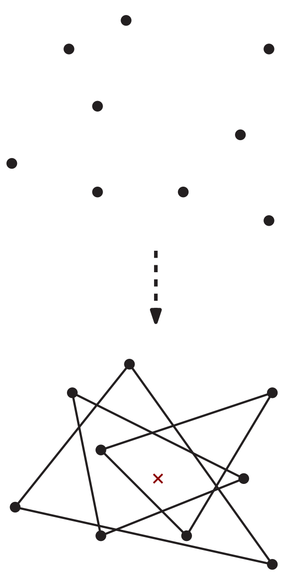

Figure 1. Any nine points in the plane can be split into three triangles that surround a common point — the red cross

Two years after the publication of Birch's paper, Norwegian graduate student Helge Tverberg proved the same result, unaware of the earlier work by Birch. After learning about Birch's theorem for the planar case and the corresponding conjecture for higher dimensions, Tverberg visited Birch in Manchester in 1964. Tverberg would later describe the weather in Manchester during his visit as "bitterly cold," and unable to sleep for the last night of his visit, since the heater in the hotel had gone off, the solution to Birch's conjecture finally dawned on Tverberg. Since then Birch's conjecture has been known as Tverberg's theorem and is now considered a central result in discrete geometry.

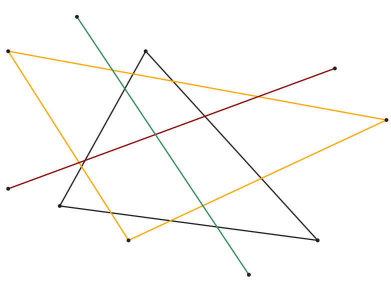

Figure 2. Ten points in the plane and a color-coded partition into four parts with intersecting convex hulls

The $n = 2$ case of Tverberg's theorem, that any $d + 2$ points in $\mathbb{R}^d$ can be split into two sets, whose convex hulls intersect, had already been established decades earlier by Johann Radon in 1921. The planar case of Radon's result states that for any four points in $\mathbb{R}^2$ either one point is contained in the triangle spanned by the other three or the four points come in two pairs such that the segments connecting the pairs intersect. Another way of saying this is that for any straight-line drawing of a tetrahedron in the plane two vertex-disjoint faces must intersect. It is natural to ask whether the drawing needs to consist of only straight lines, or whether the overlap of vertex-disjoint faces is a topological feature that holds more generally for any continuous drawing. Thus, in 1976, Imre Bárány conjectured that for any continuous map from the $(n - 1)(d + 1)$ – dimensional simplex to $\mathbb{R}^d$ there are n pairwise disjoint faces whose images have a common point of intersection. The case of an affine map exactly corresponds to Tverberg's theorem, since the image of a face under an affine map is the convex hull of its vertices.

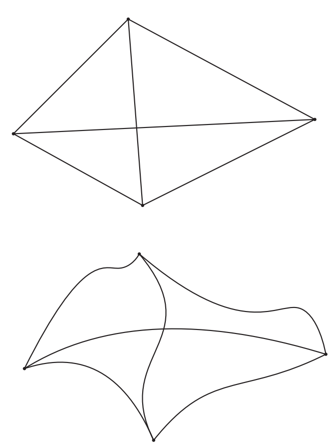

Figure 3. A linear and a continuous drawing of a tetrahedron in the plane

Bárány's conjecture became known as the topological Tverberg conjecture and was soon after — in 1981— shown to hold for n a prime. Özaydin later showed the case of n a power of a prime. This dependence on divisibility conditions of the intersection multiplicity n should come as a surprise. It stems from the following construction: If $f : X \longrightarrow \mathbb{R}^d$ avoids n-fold points of coincidence, then it induces a map from n-tuples of pairwise distinct points in X, the *configuration space* of X, to $(\mathbb{R}^d)^n$ that avoids the diagonal, that is, the subspace of $(\mathbb{R}^d)^n$, where all n factors are equal. Moreover, this map respects the symmetries of permuting coordinates. Obstructions for the existence of such *equivariant* maps depend on whether n is a prime power or not, since the symmetries have a different structure depending on the number of distinct prime divisors of n. More precisely, an elementary abelian group of symmetries acts transitively on coordinates if and only if n is a prime power.

That the topological Tverberg conjecture could only be settled in the case of n a prime power was believed to be an artifact of the proof method, which relies on lower bounds for the equivariant topology of the associated configuration space. Recent (2015) counterexamples to the topological Tverberg conjecture for every n with at least two distinct prime divisors thus were a surprise to many. The main ingredients for these counterexamples are an extension of the Whitney trick, which removes pairs of double points to construct an embedding, to higher multiplicity intersections due to Isaac Mabillard and Uli Wagner, and the "constraint method" of Pavle Blagojevic, Günter Ziegler, and the author. The former provides a way to transform vanishing obstructions for the existence of equivariant maps from the configuration space of X into maps $X \longrightarrow \mathbb{R}^d$ that avoid n–fold points — at least as long as X has codimension $\frac{d}{n}$ in $\mathbb{R}^d$.

The latter can be used to lift maps with no n-fold points in positive codimension to the case, where X may have dimension much larger than d — the case of interest for the topological Tverberg conjecture.

The constraint method proves results about the existence of n-fold points in a generalized fashion, encapsulating graph planarity, the embeddability of manifolds into $\mathbb{R}^d$, and their higher multiplicity generalizations. Perhaps surprisingly, it also exposes chromatic numbers of uniform hypergraphs as a more rigid version of the same theory.

Many seemingly simple questions remain unsolved. For instance, given three red, three green, three blue and one yellow point in the plane, can they always be split into four sets without repeated colors whose convex hulls all have a point in common?

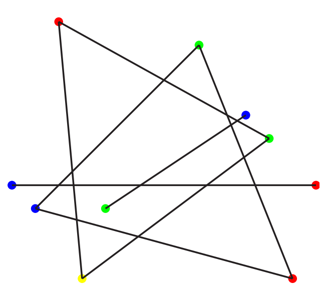

Figure 4. Three red, three green, three blue and one yellow point in the plane and a colorful 4-fold Tverberg partition.